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Lemma: (Translation Invariance)

A metric d induced by a norm on a normed space X satisfies

(a) $d(x+a, y+a) = d(x, y)$.

(b) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

Proof (a) $d(x+a, y+a) = \|(x+a) - (y+a)\|$
$= \|x - y\|$
$= d(x, y)$.

(b) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$
$= \|\alpha(x-y)\| = |\alpha| \|x-y\|$
$= |\alpha| d(x, y)$.

(1) 2.2.8 Can every metric on a vector space be obtained from a norm?

Ans: NO;

Counter example;

Define discrete metric d on X by

$d : X \times X \longrightarrow [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

clearly, d is metric on X.

Now define $d(x, y) = \|x - y\|$

Then (i) $\|x - y\| \geq 0$

(ii) $\|x - y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

but (iii) $\|\alpha(x-y)\| \neq |\alpha| \|x - y\|$.

$\because \|\alpha(x-y)\| = \|\alpha x - \alpha y\|$
$= d(\alpha x - \alpha y)$
$= |\alpha| d(x, y) = \begin{cases} 0 & \text{if } x = y \\ |\alpha| & \text{if } x \neq y \end{cases}$
$\neq |\alpha| \|x - y\|$.

Hence $\|\alpha(x-y)\| \neq |\alpha| \|x - y\|$